A BOUNDARY LAYER SOLUTION OF THE 
AXISYMMETRIC JET INSTABILITY PROBLEM 
I. ISOTHERMAL, INCOMPRESSIBLE JETS

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A boundary layer solution of the jet instability problem is carried out for realistic jet velocity profiles. The instability mode eigenvalues are determined as a power series in terms of the shear layer thickness. Comparisons with published numerical solutions show favorable accuracy for a significantly smaller computational effort. The analytical nature of the results is useful in studying developing jet flow instability modes.

1. INTRODUCTION

The instability modes of jets are of considerable practical interest in a number of applications. The knowledge of jet instability modes is essential for a proper control of devices that deposit small particles on surfaces in a precise geometric position; ink jet printers are examples. In aeroacoustic applications jet instability modes are noise source, the far field sound intensity and frequency being largely determined by the dominant instability mode [17].

Calculations of jet instability modes are carried out in a first approximation through use of the linearized equations of fluid motion. Through the usual techniques of instability theory [6] a single ordinary differential equation (ODE) is usually obtained. The eigenmodes of the equation are the sought after instability modes. The solution of this eigenproblem presents some difficulty though, mainly due to two characteristics. First, the unperturbed mean flow is usually an interpolation of measured quantities. This is usually necessary in order to obtain accurate information about instability mode characteristics such as frequency and growth rate. Secondly, the domain of the ODE is typically infinite. For studies in which the mean velocity presents discontinuities additional difficulties are encountered. The most widely used techniques to solve the ODE are numerical in nature coupled with analytical techniques such as Frobenius series expansions to avoid singularities that arise at the jet center line [9]. The eigenvalues of the problem are determined by an iterative search, a solution of the ODE being required for each step. An eigenvalue is accepted

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if the ODE's solution satisfies the proper boundary conditions on the center line and at infinity. This procedure requires significant computational resources, even more so if we take into account that there exist an infinite number of eigenvalues of the jet instability problem.

Close to the nozzle the mean jet velocity is almost constant radially except for a small zone near the jet diameter where it undergoes rapid variations, attaining the velocity of the ambient fluid. The problem thus presents a natural boundary layer. In this paper a boundary layer approach to the jet instability problem is carried out for axisymmetric incompressible jets. Results for compressible jets are presented in a companion paper. In section 1 the governing ODE equation is derived. The fluid domain is then divided into two zones: one far away from the boundary layer and the boundary layer. Far away from the boundary layer an analytical solution of the governing ODE is easily obtainable. In section 2 the boundary layer is examined using a stretching transformation. A perturbation solution of the transformed governing ODE is obtained. Matching of the two solutions leads to a hierarchy of implicit equations for the eigenvalues of the jet instability problem in terms of powers of the boundary layer thickness. This is presented in section 3. The two solutions may also be joined by the elementary procedure of patching. This also is presented in section 3. Practical examples of the matching procedure for several typical mean velocity distributions are presented in section 4. For the first few orders of the power series expansion closed form analytical results are obtained. To the author's knowledge these are the first analytical results published. The analytical form of the results is especially useful for developing jet instability studies as presented in the third paper in this series. Examples of the patching procedure are presented in section 5. Calculations and comparisons with the standard numerical procedure are carried out in section 6. Conclusions are presented in section 7.

2. THE JET INSTABILITY EIGENPROBLEM

The problem to be considered is that of a jet coming out of a nozzle and entering into a motionless fluid medium (fig. 1). The jet fluid and the fluid medium are the same though they may have different thermodynamic states. We introduce a cylindrical coordinate system and assume that the mean jet velocity \( \overline{V} \) has no radial or tangential components \( \overline{V} = (\overline{U(r)},0,0) \). The jet radius \( R \) is defined by \( \overline{U}(R) = U_j / 2 \) where \( U_j \) is the jet centerline velocity. Non-dimensional flow variables are introduced using \( U_j, R \) and the jet centerline temperature \( T_j \) as reference values. Dimensional quantities shall generally be denoted with an overbar henceforth. The non-dimensional velocity is \( U(r) = \overline{U(r)} / U_j \) with the obvious particular values \( U(0) = 1, U(1) = 1 / 2 \). We assume the mean jet pressure is constant inside the jet and equal to the ambient pressure so as to ensure a parallel flow. Since the jet is a shearing flow of small transversal extent standard boundary layer considerations are applicable and assuming that the Prandtl number is unity
(e.g. for air) the Busemann-Crocco temperature law is valid \[16\]

\[ T(r) = \frac{T_\infty}{T_j} = \frac{T_\infty}{T_j} + \left(1 - \frac{T_\infty}{T_j}\right) U(r) + \frac{1}{2} M^2 U(r) \left[1 - U(r)\right] \]  

(1)

where \(T_\infty\) is the dimensional fluid temperature at infinite radial distance from the jet axis and \(M = U_j/\alpha_j\) is the jet center line Mach number.

Realistic velocity profiles, as measured experimentally, show significant velocity variations mainly in the small jet shear layer at \(r \approx 1\). The non-dimensional shear layer thickness \(\theta\) is introduced as

\[ \theta = \int_0^r \rho(r) U(r)' [1 - U(r)'] dr \]

with \(\rho(r)\) the non-dimensional fluid density. Some typical analytical approximations to measured velocity profiles are listed below along with the relevant authors who used these profiles. A graphical presentation of the various velocity profiles is given in fig. 2. The first four profiles are typical for a jet just emerging from a
nozzle. The last one (6), shows the velocity profile at a large downstream distance.
- Troutt & McLaughlin [18]

\[ U(r) = \begin{cases} 
\exp\left[-n\left[b_1(r-1)/\theta+1\right]^2\right], & r > 1 - \theta / 2, \quad b_1 = 2 \\
1, & r \leq 1 - \theta / 2
\end{cases}, \quad n = \ln 2 \tag{2} \]

- Michalke [9], Morris [13]

\[ U(r) = 0.5 \cdot \left[1 + \tanh\left[b_2(1-r)/\theta\right]\right], \quad b_2 = 0.5 \tag{3} \]


\[ U(r) = 0.5 \cdot \left[1 + \tanh\left[b_3(1-r)/\theta\right]\right], \quad b_3 = 0.25 \tag{4} \]

- Mattingly & Chang [8]

\[ U(r) = \exp\left[n\left[b_4(r-1)/\theta+1\right]^2\right], \quad b_4 = 0.312, \quad n = \begin{cases} 
0, & r \leq 1 - \theta / b_4 \\
-\ln 2, & r > 1 - \theta / b_4
\end{cases} \tag{5} \]

- Lessen & Singh [7], Mollendorf & Gebhart [12], Morris [13].

\[ U(r^2 - 1)^2 \cdot b_5 = \sqrt{2 - 1} = 2.185 \tag{6} \]

The above mean flows shall be used in an ODE characterizing the jet's linear instability. This equation is derived from the Euler equations of fluid motion coupled with an entropy transport equation.

\[ \frac{\partial \tilde{\rho}}{\partial t} + \nabla(\tilde{\rho} \vec{V}) = 0 \]

\[ \frac{\partial \vec{V}}{\partial t} = -\nabla \tilde{p} \]

\[ \frac{\partial \tilde{\rho}}{\partial t} = 0. \]

A cylindrical coordinate system is introduced. The velocity perturbations to the mean flow are denoted by \((u(x, r, \phi, t), v(x, r, \phi, t), w(x, r, \phi, t))\) and the perturbations in thermodynamic parameters are \((p'(x, r, \phi, t), \rho'(x, r, \phi, t), T'(x, r, \phi, t), s'(x, r, \phi, t))\)
with the usual notations. The equations are linearized around the mean flow by separating instantaneous quantities into means and perturbations \( \bar{\rho}(x, r, \phi, t) = \rho(r) + \rho'(x, r, \phi, t) \), with analogous expressions for the other field variables. The linearized equations of motion are

\[
\frac{\partial \rho'}{\partial t} + U \frac{\partial \rho'}{\partial x} = -\rho \left( \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial r} + \frac{v'}{r} + \frac{1}{r} \frac{\partial w'}{\partial \phi} \right) - v' \frac{\partial p}{\partial r}
\]

\[
\rho \left( \frac{\partial u'}{\partial t} + U \frac{\partial u'}{\partial x} + v' \frac{\partial U}{\partial x} \right) = -\frac{\partial p'}{\partial x}
\]

\[
\rho \left( \frac{\partial v'}{\partial t} + U \frac{\partial v'}{\partial x} + v' \frac{\partial U}{\partial r} \right) = -\frac{\partial p'}{\partial r}
\]

\[
\rho \left( \frac{\partial w'}{\partial t} + U \frac{\partial w'}{\partial x} + v' \frac{\partial U}{\partial \phi} \right) = -\frac{1}{r} \frac{\partial p'}{\partial \phi}
\]

\[
\frac{\partial s'}{\partial t} + U \frac{\partial s'}{\partial x} + v' \frac{ds}{dr} = 0
\]

For a perfect gas the entropy equation may be rewritten in terms of pressure and density

\[
\bar{s} - s_0 = c_v \ln \frac{\bar{\rho}}{\rho_0} + c_p \ln \frac{\rho_0}{\rho}
\]

where \((\rho_0, s_0)\) is some thermodynamic reference state. Introducing the mean flow-perturbation separation \( \bar{s} = s + s', \bar{\rho} = p + p', \bar{\rho} = \rho + \rho' \) we have

\[
s + s' - s_0 = c_v \ln \frac{\rho + \rho'}{\rho_0} + c_p \ln \frac{\rho_0}{\rho + \rho'}
\]

Subtracting the relationship for mean flow quantities from the above equation leads to

\[
s' = c_v \ln \left( 1 + \frac{\rho'}{\rho} \right) - c_p \ln \left( 1 + \frac{\rho'}{\rho} \right)
\]

Using the series expansion of \( \ln \) the entropy transport equation may be rewritten in terms of pressure and density as

\[
\frac{\partial p'}{\partial t} + U \frac{\partial p'}{\partial x} = \frac{1}{k R_g T} \left( \frac{\partial p'}{\partial t} + U \frac{\partial p'}{\partial x} \right) - v' \frac{\partial p}{\partial r}
\]

where \( R_g \) is the gas constant.
For the above linearized equations of motion we use the method of normal modes [6]

\[ (u', v', w', p', \rho') \sim (\hat{u}(r), \hat{v}(r), \hat{w}(r), \hat{p}(r), \hat{\rho}(r)) \exp[i(\alpha x + m \phi - \omega t)] \]

and obtain the equations

\[ i\alpha(U - c)\hat{\rho} = -\rho \left( i\alpha \hat{u} + \frac{d\hat{v}}{dr} + \frac{\hat{v}}{r} + \frac{im}{r} \hat{w} \right) - \hat{v} \frac{d\hat{p}}{dr} \]

\[ i\alpha \rho(U - c)\hat{\omega} + \rho \hat{v} \frac{dU}{dr} = -i\alpha \hat{\rho} \tag{7} \]

\[ i\alpha(U - c)\hat{\nu} = -i\alpha \frac{d\hat{\rho}}{dr} \]

\[ i\alpha(U - c)\hat{\rho} = i\alpha M^2(U - c)\hat{\rho} \]

where \( c = \omega/\alpha \) is the normal mode phase velocity. Elimination of \( \hat{u}, \hat{v} \) and \( \hat{\rho} \) leads to a single second order ODE

\[ \frac{d^2\hat{\rho}}{dr^2} + \left( 1 - \frac{1}{W} \frac{dW}{dr} \right) \frac{d\hat{\rho}}{dr} - \left[ \alpha^2(1 - M^2W) + \frac{m^2}{r^2} \right] \hat{\rho} = 0 \tag{8} \]

where the influence of the mean flow has been isolated into the single quantity

\[ W(r) = \frac{(U - c)^2}{T}. \tag{9} \]

Appropriate boundary conditions are \( \hat{\rho}(r \to \infty) \to 0 \) and \( \hat{\rho}(0) \) finite.

Taking note of the typical mean flow velocity profiles one may observe that at \( r \sim 0 \) the velocity has an almost constant value \( U(r \sim 0) \approx 1 \) and also at \( r \to \infty \) the velocity is constant \( U(r \to \infty) \approx 0 \). Since the temperature is directly related to the velocity distribution by \( (1) \) \( \frac{dW}{dr} \) is almost zero at the jet axis and at infinity. Equation (8) then has the following asymptotic forms

\[ r \to 0: \frac{d^2\hat{\rho}}{dr^2} + \frac{1}{r} \frac{d\hat{\rho}}{dr} - \left[ \alpha^2(1 - M^2W(0)) + \frac{m^2}{r^2} \right] \hat{\rho} = 0 \tag{10} \]

\[ r \to \infty: \frac{d^2\hat{\rho}}{dr^2} + \frac{1}{r} \frac{d\hat{\rho}}{dr} - \left[ \alpha^2(1 - M^2W(\infty)) + \frac{m^2}{r^2} \right] \hat{\rho} = 0 \tag{11} \]

both of these being modified Bessel equations. The modified Bessel functions satisfy \( I_n(r \to \infty) \to \infty, K_n(r \to 0) \to \infty \) the appropriate general form of the solutions to (10), (11) is...
At \( r \sim 1 \) the two asymptotic solutions should match. This is not generally possible for arbitrary values of \( \alpha \) and therefore an eigenproblem for \( \alpha \) has been established. The standard method [3] for determining the eigenvalues is to numerically solve (8) for some chosen pair \((\alpha, \omega)\) and see if the correct matching at \( r \sim 1 \) is obtained. If not, another pair \((\alpha, \omega)\) is chosen according to some search procedure. This program has been carried out for (8) by Michalke [9]. For especially simple forms of the function \( W \), closed form solutions may be obtained though. An example is when the velocity has the profile known as "plug flow"

\[
U(r) = \begin{cases}
1 & , r < 1 \\
1/2 & , r = 1 \\
0 & , r > 1
\end{cases}
\]

In this case the jet interior is separated from the ambient fluid by a vortex sheet. Let \( \zeta(x, \phi, t) \) be the radial displacement of the vortex sheet. Matching of the pressure specified by (10), (11) at \( r = 1 \) for an isothermal \( T_a/T_j = 1 \), incompressible \( M = 0 \) jet leads to

\[
\hat{p}_{\text{int}}(1) = \hat{p}_{\text{ext}}(1) \Rightarrow A I_n(\alpha) = B K_n(\alpha).
\]  

(13)

The radial velocity \( v \) satisfies the equation

\[
v = \frac{D \zeta}{Dt}
\]

which leads to the following relationships between Fourier components

\[
\hat{v}_{\text{int}}(1) = i \alpha (1-c) \hat{\zeta}, \quad \hat{v}_{\text{ext}}(1) = i \alpha (-c) \hat{\zeta}.
\]

Since

\[
\hat{v}_{\text{int}}(1) = -\frac{1}{1-c} \frac{d \hat{p}_{\text{int}}}{dr} = -\frac{A I_n'(\alpha)}{1-c}, \quad \hat{v}_{\text{ext}}(1) = \frac{1}{c} \frac{d \hat{p}_{\text{ext}}}{dr} = \frac{B K_n'(\alpha)}{c}
\]

matching of the radial displacement leads to the following relationship between the integration constants \( A, B \)

\[
\frac{A}{B} = \left( \frac{1-c}{c} \right)^2 \frac{K_n(\alpha)}{I_n'(\alpha)}
\]  

(14)
Combining this with the pressure matching condition the following characteristic equation, first obtained by Batchelor & Gill [1] is deduced

\[
\left(1 - \frac{c}{c_c}\right)^2 = \frac{K_m(\alpha) I'_m(\alpha)}{K'_m(\alpha) I_m(\alpha)}.
\] (15)

For a compressible jet the same procedure leads to a more complicated characteristic equation

\[
\frac{T_\infty}{T_j} \left(1 - \frac{c}{c_c}\right)^2 = \frac{a_0}{a_\infty} \frac{K_m(a_\infty, \alpha) I'_m(a_\infty, \alpha)}{K'_m(a_\infty, \alpha) I_m(a_\infty, \alpha)}
\]

where

\[
a_0 = \sqrt{1 - M^2 W(0)}, \quad a_\infty = \sqrt{1 - M^2 W(\infty)}.
\]

3. THE BOUNDARY LAYER SOLUTION OF THE JET INSTABILITY PROBLEM

3.1. THE EQUATION HIERARCHY

Consider now how the jet interior solution from (12) changes into the jet exterior solution from (12) by passing through the boundary layer. In order to analyze this a stretching transformation

\[
s = \frac{r - 1}{\theta}
\]

is introduced. This transformation has the effect of magnifying the region around \( r = 1 \). The power expansions

\[
\frac{1}{r} \quad \frac{1}{r^2} = 1 - s \theta + s^2 \theta^2 - ...
\]

\[
c = c_0 + \theta c_1 + \theta^2 c_2 + ...
\]

\[
U(s) = U_0(s) + \theta U_1(s) + \theta^2 U_2(s) + ...
\]

\[
\beta(s) = P_0(s) + \theta P_1(s) + \theta^2 P_2(s) + ...
\] (16)

\[
\beta(s) = \alpha \sqrt{1 - M^2 W(s)} = \beta_0(s) + \theta \beta_1(s) + \theta^2 \beta_2(s) + ...
\]
\[ W(s) = W_0(s) + \theta W_1(s) + \theta^2 W_2(s) + \ldots \]
\[ \ln W(s) = \ln W_0(s) + \theta \Omega_1(s) + \theta^2 \Omega_2(s) + \ldots \]
\[ \Omega_1(s) = 2 \frac{U_1 - c_1}{U_0 - c_0} - \frac{T_1}{T_0}, \quad \Omega_2(s) = 2 \frac{U_2 - c_2}{U_0 - c_0} - \frac{(U_1 - c_1)^2}{(U_0 - c_0)^2} + \frac{1}{2} \frac{T_1^2}{T_0^2} - \frac{T_2}{T_0} \]

are introduced in equation (8). By identifying the powers of \( \theta \) the following hierarchy of equations is obtained

\[ \mathcal{O}(1): \frac{d^2 P_0}{ds^2} - \left( \frac{d}{ds} \ln W_0 \right) \frac{dP_0}{ds} = 0 \]
\[ \mathcal{O}(\theta): \frac{d^2 P_1}{ds^2} - \left( \frac{d}{ds} \ln W_0 \right) \frac{dP_1}{ds} = \frac{dP_0}{ds} f_1 \]
\[ \mathcal{O}(\theta^2): \frac{d^2 P_2}{ds^2} - \left( \frac{d}{ds} \ln W_0 \right) \frac{dP_2}{ds} = \frac{dP_0}{ds} f_2 + \frac{dP_1}{ds} f_1 + (\beta_0^2 + m^2) P_0 \]

where

\[ W_0(s) = \frac{(U_0 - c_0)^2}{T_0}, \quad f_1(s) = \frac{d\Omega_1}{ds} - 1, \quad f_2(s) = s + \frac{d\Omega_2}{ds} \]

Successive terms in the pressure perturbation expansion may be determined by solving the above hierarchy. The boundary conditions for the above equations are to be determined by matching with the pressure inside and outside the jet in the vicinity of the shear layer. The \( \mathcal{O}(1) \) ODE is easily solvable

\[ P_0(s) = C_0 W_0(s) + D_0, \quad W_0(s) = \int W_0(s) ds. \]

The other equations in the hierarchy are inhomogeneous versions of the first, \( \mathcal{O}(1) \) equation. They may be solved by the method of variation of constants [2, 1.5]. The general solution of the homogeneous part of the equation is

\[ P_k(s) = F_k W_1(s) - G_k \cdot 1 \]

where \( F_k, G_k \) are constants of integration. The two fundamental solutions \( W_0(s) \) and \( 1 \) have the Wronskian

\[
\begin{vmatrix}
W_i(s) & 1 \\
W_0(s) & 0
\end{vmatrix} = -W_0(s).
\]
If \( h_k(s) \) is the inhomogeneous part (right hand side) then the functions \( F_k(s) \), \( G_k(s) \) are determined from

\[
F'_k(s) = \frac{h_k(s)}{W_0(s)}, \quad G'_k(s) = -\frac{h_k(s)W_i(s)}{W_0(s)}
\]  

(19)

For the \( O(\theta) \) equation

\[
h_0(s) = F_0'(s)f_0(s) = C_0W_0(s)f_0(s),
\]

\[
F_i(s) = C_0f_i(s) \Rightarrow F_i(s) = C_0\int f_i(s)ds + C_1,
\]

\[
G_i(s) = -C_0f_i(s)W_i(s) \Rightarrow G_i(s) = -C_0\int f_i(s)W_i(s)ds + D_1
\]

so the solution is

\[
P_i(s) = \left[C_0\int f_i(s)ds + C_1\right]W_i(s) - C_0\int f_i(s)W_i(s)ds + D_1.
\]  

(20)

The procedure may be continued to any order.

### 3.2. THE MATCHING PROCEDURE

We must now match the 'inner solutions which are valid in the boundary layer with the outer solutions given by (12). We use the general matching principle: the inner expansion of the outer solution = the outer expansion of the inner solution. Since there are two outer expansions matching has to be carried out both for the jet interior and jet exterior. We shall carry out the matching procedure to \( O(\theta^2) \) using Van Dyke's matching principle as presented in [14, 24].

**Jet interior matching.** The outer solution is

\[
\tilde{\beta}_{\text{int}}(r) = AI_m'\left(\beta_0 + \beta_1s + \beta_2s^2 + \ldots\right) r.
\]

The inner region (boundary layer) is at \( r \equiv 1 \). The three term outer expansion

\[
AI_m(\beta_0 r) + AI_m'(\beta_0 r)\theta + \left[\frac{1}{2} r^2 A^2 \beta_0^2 I'_m(\beta_0 r) + \beta_1 r^2 I'_m(\beta_0 r)\right] \theta^2 + ...
\]

rewritten in the inner variable \( s \) and expanded for small \( \theta \)

\[
AI_m(\beta_0) + AI_m'(\beta_0)(\beta_1 + \beta_2 s)\theta + \left[\frac{1}{2} I'_m(\beta_0)(\beta_2 + \beta_1 s) + \frac{1}{2} I'_m''(\beta_0)(\beta_1 + \beta_2 s)\right] \theta^2 + ...
\]
leads to the following terms in the matching variable $r - 1$

\[ O(1): \quad A \left[ I_m (\beta_0) + \beta_0 (r - 1) I'_m (\beta_0) + \frac{1}{2} \beta_0^2 (r - 1)^2 I''_m (\beta_0) \right] \]

\[ O(\Theta): \quad A \left[ \beta_1 I'_m (\beta_0) + \beta_1 (r - 1) I'_m (\beta_0) + \beta_0 \beta_1 (r - 1) I''_m (\beta_0) \right] \]  

\[ O(\Theta^2): \quad A \left[ \beta_2 I'_m (\beta_0) + \frac{1}{2} \beta_2^2 I''_m (\beta_0) \right] \]  

Matches to lesser orders may be obtained from the above by eliminating cross diagonal terms. For $O(\Theta)$ matching we would use

\[ O(1): \quad A \left[ I_m (\beta_0) + \beta_0 (r - 1) I'_m (\beta_0) \right] \]

\[ O(\Theta): \quad A \beta_1 I'_m (\beta_0) \]  

while for $O(1)$ matching we would use

\[ O(1): \quad A I_m (\beta_0) \]  

The inner solution is $\hat{p}(s) = P_0(s) + \Theta P_1(s) + \Theta^2 P_2(s) + \ldots$. The outer region of this inner solution is at $s \rightarrow -\infty$. The asymptotic behavior of the inner solution as $s \rightarrow -\infty$ depends on the chosen mean velocity field through $\mathcal{W}(s)$. After obtaining this asymptotic behavior for a given function $\mathcal{W}(s)$, matching may be carried out in an intermediate region which overlaps both the inner region and the outer region. An appropriate intermediate region is given by $1 - r \sim \Theta^{1/2}$ for which $s \sim -\Theta^{-1/2}$. In this region corresponding powers of $1 - r$ are identified in the two expansions. Examples are given below.

**Jet exterior matching.** The outer solution is

\[ \hat{p}_{\text{ext}}(r) = B K_m (\beta_0 + \beta_1 \Theta + \beta_2 \Theta^2 + \ldots) r \]

The procedure outlined above leads to a similar result

\[ O(1): \quad B \left[ K_m (\beta_0) + \beta_0 (r - 1) K'_m (\beta_0) + \frac{1}{2} \beta_0^2 (r - 1)^2 K''_m (\beta_0) \right] \]

\[ O(\Theta): \quad B \left[ \beta_1 K'_m (\beta_0) + \beta_1 (r - 1) K'_m (\beta_0) + \beta_0 \beta_1 (r - 1) K''_m (\beta_0) \right] \]  

\[ O(\Theta^2): \quad B \left[ \beta_2 K'_m (\beta_0) + \frac{1}{2} \beta_2^2 K''_m (\beta_0) \right] \]  

Matching is carried out as above in an intermediate region where $r - 1 \sim \Theta^{1/2}$ and $s \sim \Theta^{-1/2}$.

The inner solution has to be the same in both cases. This is generally not possible for arbitrary values of $\beta$. The restrictions which have to be placed on $\beta$ are the characteristic relations for a jet with a non-zero shear layer thickness.
3.3. SOLUTION PATCHING

Another procedure of joining together the jet interior and exterior solution is to impose the same values of the pressure and/or its derivatives at specific points regarded as the transition between the shear layer and the mean flow. This procedure is known as patching [2]. Let \( r = 1 - n\theta \) be the separation point between the jet interior and the shear layer while \( n \) is a yet undetermined constant. Similarly, \( r = 1 + n\theta \) is the separation point between the jet exterior and the shear layer. The first few patching conditions are

\[
\begin{align*}
AI_m[\beta_0(1 - n\theta)] &= P_0(-n) + \theta P_1(-n) + \\
BK_m[\beta_\infty(1 + n\theta)] &= P_0(n) + \theta P_1(n) + \\
AB_\infty K_m[\beta_0(1 - n\theta)] &= \frac{1}{\theta} [P_0'(-n) + \theta P_1'(-n) + \\
BB_\infty K_m[\beta_\infty(1 + n\theta)] &= \frac{1}{\theta} [P_0'(n) + \theta P_1'(n) + 
\end{align*}
\]  

(25)

with \( \beta_0 = \alpha a_0, \beta_\infty = \alpha a_\infty \). These conditions likewise are not generally satisfied for arbitrary \( \alpha \). The conditions on \( \alpha \) under which the conditions hold are the dispersion relations. Though this procedure would seem to be superseded by solution matching it shall be shown in section 6 that it is also useful.

4. THE MATCHING PROCEDURE FOR TYPICAL VELOCITY PROFILES

4.1. INCOMPRESSIBLE, ISOTHERMAL JET WITH TANH VELOCITY DISTRIBUTION

We start with mean velocity profile (3) for which useful analytical results may be obtained. The expansion of \( U(s) \) in powers of \( \theta \) has only one term \( U(s) = U_0(s) \). Consider first an incompressible \( (M = 0) \) isothermal \( (T_o/T_j = 1) \) jet for which \( \beta = \alpha \). In this case

\[
W_0(s) = \left[ \frac{1}{2} \left( 1 - \tanh \frac{s}{2} \right) - c_0 \right]^2
\]

and

\[
Wi(s) = \int W_0(s) ds = \frac{1}{2} \frac{2c_0 + 2c_0^2}{2} s + (2c_0 - 1) \ln \cosh \frac{s}{2} - \frac{1}{2} \tan \frac{s}{2}. \tag{26}
\]

The \( O(1) \) inner solution is \( P_0(s) = C_0 Wi(s) + D_0 \). We first carry out the matching to this order.
\( O(1) \) jet interior matching. The asymptotic behavior of (26) as \( s \to \infty \) is given by

\[
\Psi_i(s) \sim (1 - c_0)^2 s + \frac{1}{2} - (2c_0 - 1) \ln 2
\]

and the resulting outer expansion of the inner solution is

\[
P_o = C_0 \left[ (1 - c_0)^2 \frac{r - 1}{\theta} + \frac{1}{2} - (2c_0 - 1) \ln 2 \right] + D_0.
\]

We identify powers of \( r - 1 \) from the above with (23) to \( O(1) \). The following system of equations is obtained

\[
\begin{align*}
\left[ \frac{1}{2} - (2c_0 - 1) \ln 2 \right] C_0 + D_0 &= AI_m(\beta_0) \\
\frac{1}{\theta} C_0 (1 - c_0)^2 &= 0
\end{align*}
\] (27)

\( O(1) \) jet exterior matching. The asymptotic behavior of (26) as \( s \to \infty \) is given by

\[
\Psi_i(s) \sim c_0^2 s - \frac{1}{2} - (2c_0 - 1) \ln 2
\]

and the outer expansion of the inner solution is

\[
P_o = C_0 \left[ c_0^2 \frac{r - 1}{\theta} - \frac{1}{2} - (2c_0 - 1) \ln 2 \right] + D_0.
\]

On comparing like powers of \( r - 1 \) with (24) the following system of equations is obtained

\[
\begin{align*}
\left[ -\frac{1}{2} - (2c_0 - 1) \ln 2 \right] C_0 + D_0 &= BK_m(\beta_0) \\
\frac{1}{\theta} C_0 c_0^2 &= 0
\end{align*}
\] (28)

\( O(1) \) eigenvalue condition. Both (27) and (28) imply \( C_0 = 0 \). Since the constant \( D_0 \) is the same at both boundaries of the inner region we must have

\[
AI_m(\beta_0) = BK_m(\beta_0).
\]

We conclude that \( O(1) \) matching leads to the physical condition of pressure continuity across a vortex sheet obtained earlier from physical arguments (13).
The above equation cannot determine the eigenvalue since the constants $A$, $B$ are not specified. Eigenmode normalization permits choice of one of them but not both. We shall choose $A = 1$ henceforth. To obtain another equation from which we can calculate the eigenvalue we continue the matching procedure to $O(\theta)$. The $O(\theta)$ inner solution is $P_0(s) + \theta P_\tau(s) = D_0 + \theta[C_1 W(s) + D_1]$. From the $O(1)$ match we already know the asymptotic behavior of this solution.

$O(\theta)$ jet interior matching. The outer expansion of the inner solution is

$$P_0 + \theta P_\tau = D_0 + \theta C_1 \left[ (1-c_0)^2 \frac{r-1}{\theta} + \frac{1}{2} - (2c_0 - 1)\ln 2 \right] + \theta D_1$$

We identify powers of $r-1$ from the above with (22) to $O(\theta)$ and obtain

$$D_0 = AI_m(\beta_0)$$

$$C_1 (1-c_0)^2 = \beta_0 AI'_m(\beta_0)$$

$$\left[ \frac{1}{2} - (2c_0 - 1)\ln 2 \right] C_1 + D_1 = \beta_1 AI'_m(\beta_0)$$

(29)

$O(\theta)$ jet exterior matching. A similar procedure with

$$P_0 + \theta P_\tau = D_0 + \theta C_1 \left[ c_0^2 \frac{r-1}{\theta} - \frac{1}{2} - (2c_0 - 1)\ln 2 \right] + \theta D_1$$

leads to the following jet exterior matching equations

$$D_0 = BK_m(\beta_0)$$

$$C_1 c_0^2 = \beta_0 BK'_m(\beta_0)$$

$$\left[ - \frac{1}{2} - (2c_0 - 1)\ln 2 \right] C_1 + D_1 = \beta_1 BK'_m(\beta_0)$$

(30)

$O(\theta)$ eigenvalue condition. Since $C_1$ has to be the same in (29) and (30) we obtain

$$\frac{(1-c_0)^2}{c_0^2} = \frac{A}{B} \frac{I'_m(\beta_0)}{K'_m(\beta_0)}$$

the vortex sheet displacement continuity condition (14). It is now possible to eliminate $A/B$ and obtain the vortex sheet dispersion relationship (15). We also obtain an $O(\theta)$ correction to the eigenvalue

$$\beta_1 = \frac{\beta_0}{1 - 2c_0}.$$
The \( O(\Theta^2) \) equation is now

\[
\frac{d^2 P_2}{ds^2} - \left( \frac{d}{ds} \ln W_0 \right) \frac{dP_2}{ds} = D_{0m} + C_1 f_1(s) W_0(s) = h_2(s)
\]

with \( D_{0m} = D_0 B_0^2 + m^2 \) so the functions \( F_2(s) \), \( G_2(s) \) resulting from (19) are

\[
F_2(s) = -C_1 \left( \frac{2 c_1}{U - c_0} + s \right) + D_{0m} \int \frac{ds}{W_0(s)} + C_2
\]

\[
G_2(s) = -D_{0m} \int \frac{W_1(s)}{W_0(s)} ds - C_1 \int f_1(s) W_1(s) ds + D_2
\]

(32)

and \( P_2(s) \) is

\[
P_2(s) = F_2(s) \text{Wi}(s) + G_2(s).
\]

Let \( c_r = c_0 \), \( c_0 = 1 - c_0 \) be the limiting velocities as \( s \to \infty \), \( s \to -\infty \) respectively. The asymptotic behavior for \( F_2(s) \) is

\[
F_2(s) \sim \left( -C_1 + \frac{D_{0m}}{c_0^2} \right) s + F_{2,0}^\pm + C_2 \quad s \to \pm \infty.
\]

(33)

with

\[
F_{2,0}^\pm = \pm \frac{c_1}{c_2} C_1 + \frac{D_{0m}}{c_0 (1 - c_0) c_r} \left[ \frac{1 - 2 c_0}{c_r} \ln(\pm c_r) - \frac{1}{1 - 2 c_0} \right]
\]

The asymptotic behavior for \( G_2(s) \) may be determined by repeated use of l'Hôpital's rule. The leading order, \( O(s^0) \) behavior is first obtained and then subtracted from \( G_2(s) \) so as to obtain the \( O(s) \) behavior. Note that the \( O(1) \) asymptotic behavior of \( G_2(s) \) is generally different at the two boundaries of the shear layer so it may not be included in the integration constant \( D_2 \), since this must have the same value across the shear layer. \( O(1) \) asymptotic behavior of \( G_2 \) can only arise from integrands that have a null limiting value as \( s \to \infty \). The only part of \( G_2 \) satisfying this criterion is denoted by \( (*) \) in (32). Analysis of this term completes the asymptotic behavior study whereupon we obtain

\[
G_2(s) - G_{2,2}^\pm s^2 + G_{2,1}^\pm s + G_{2,0}^\pm + D_2 \quad s \to \pm \infty
\]

(34)

with

\[
G_{2,2}^\pm = \frac{1}{2} \left[ C_1 c_2^2 - D_{0m} \right]
\]

\[
G_{2,1}^\pm = \pm \frac{1}{2} (2 c_0 - 1) \ln 2 \left[ \frac{D_{0m}}{c_2} - C_1 \right]
\]
\[
G_{2,0}^+ = -c_0 C_1 \left[ \pm \frac{2c_0 - 1}{c_z} + \frac{2 \ln 2}{c_z} \right]
\]

Using (33) and (34) we can determine the asymptotic behavior of \( P_2(s) \) and carry out the matching process.

**O(\( \theta^2 \)) jet interior matching.** The \( O(\theta) \) match equations (29) are obtained here again. In addition to these we also obtain

\[
-(1 - c_0)^2 C_1 + D_{0m} = \beta_0^2 A'_{m}''(\beta_0)
\]

\[
-2c_1(1 - c_0)C_1 + (1 - c_0)^2 C_2 + D_{0m} \delta_f = \beta_1 A'_{m}(\beta_0) + \beta_0 \beta_0 A''_{m}(\beta_0)
\]

\[
-2c_1(1 + 2 \ln 2)c_1 + D_2 = \left[ \frac{1}{2} + (1 - 2c_0) \ln 2 \right] C_2 + \frac{D_{0m}}{(1 - c_0)^2} \delta_f \left[ \frac{1}{2} + (1 - 2c_0) \ln 2 \right] = \beta_2 A''_{m}(\beta_0) + \frac{1}{2} \beta_0^2 A'''_{m}(\beta_0)
\]

with

\[
\delta_f = \frac{1}{c_0^2} \left[ (1 - 2c_0) \ln(c_0) - \frac{1 - c_0}{1 - 2c_0} \right]
\]

**O(\( \theta^2 \)) jet exterior matching.** In addition to the equations (30) we obtain

\[
-c_0^2 C_1 + D_{0m} = \beta_0^2 B K''_{m}(\beta_0)
\]

\[
2c_1c_0 C_1 + c_0^3 C_2 + D_{0m} = \beta_0 B K'_{m}(\beta_0) + \beta_0 \beta_0 B K''_{m}(\beta_0)
\]

\[
-2c_1(1 + 2 \ln 2)c_1 + D_2 = \left[ \frac{1}{2} + (1 - 2c_0) \ln 2 \right] C_2 + \frac{D_{0m}}{c_0^2} \left[ \frac{1}{2} + (1 - 2c_0) \ln 2 \right] = \beta_2 B K''_{m}(\beta_0) + \frac{1}{2} \beta_0^2 B K'''_{m}(\beta_0)
\]

with

\[
\delta_K = \frac{1}{(1 - c_0)^2} \left[ (1 - 2c_0) \ln(1 - c_0) - \frac{c_0}{1 - 2c_0} \right]
\]

**O(\( \theta^2 \)) eigenvalue condition.** The above equations form an overdetermined, homogeneous system of 12 equations for 7 unknowns: \( A, B, D_0, C_1, C_2, D_2 \). We find the second order correction to the eigenvalue by reducing the above
overdetermined system of equations to row-echelon form and imposing conditions on $c_1$ and $\beta_2$ so as to ensure all 7 by 7 minors have null determinants. The calculations are greatly aided by use of modern symbolic software packages [19]. The two conditions that follow allow non-zero solutions to the homogeneous linear system and are eigenvalue conditions

$$c_1 = \frac{\beta_2^2}{2c_1 c_{\infty}} \left[ c^2 I''_n K_m'' - c^2 I''_n K_m \right] - \left( \frac{\beta_2^2 + m^2(1 - 2c_0)}{2c_1 c_{\infty}} \right) \left[ (1 - 2c_0) \ln \frac{c_0}{c_{\infty}} - 1 \right] I''_n K_m$$

$$c_{\infty} = \frac{\beta_2^2}{\beta_0} \left[ c_1 I''_n K_m + I''_n K_m \right] + \left( \frac{1}{2} + 2 \right) I''_n K_m$$

$$I''_n K_m (\beta_2^2 + m^2(1 - 2c_0)) \left[ \frac{c_0}{c_{\infty}} \left( \frac{1}{2} + \ln 2 \right) \beta_1 + \frac{c_1}{c_{\infty}} \left( \frac{1}{2} - \ln 2 \right) \beta_1 \right]$$

(35)

All modified Bessel functions are evaluated at $\beta_0$ (i.e. $I''_n$ at $\beta_0$, etc.).

Note that a condition has to be imposed on $c_1$ in order for non-zero eigenmodes to exist. This was to be expected. The $O(\theta)$ homogeneous ODE for $P(\theta)$ admitted a non-zero solution. The $O(\theta^2)$ equation has the same differential operator but is inhomogeneous. In order for this equation to admit a solution a solvability condition [14] has to be imposed on the inhomogeneous term $H_n(\theta)$. The solvability condition may be determined by the general method presented in [14, 15.4] using solutions of the adjoint equation. Difficulties arise though in choosing adequate boundary conditions for the adjoint problem. Such boundary conditions can only be determined by matching with the exterior solution the procedure thus reducing itself to that presented in this paper.

As the jet shear layer thickens the above procedure finds corrections to the eigenmodes by simultaneously modifying $\alpha$, $\omega$, $c$. Since $c = \omega/\alpha$ we have

$$\omega_0 = \alpha_0 c_0$$

$$\omega_1 = \alpha_1 c_0 + \alpha_0 c_1$$

$$\omega_2 = \alpha_2 c_0 + \alpha_1 c_1 + \alpha_0 c_2$$

This contrasts somewhat with the standard numerical procedure [3] where calculations are carried out for a fixed $\omega$ typically. In the above procedure the eigenmodes that correspond to a given frequency are calculated by choosing appropriate initial $O(1)$ values for $\omega_0$ and $c_0$. Finally note that a complete $O(\theta^3)$
correction requires $c_0$. This may be obtained by imposing solvability conditions on the $O(\theta^3)$ equation in the hierarchy (17) but the calculation is not carried out here.

The mean velocity profile (3) used in the above calculations does not accurately describe jets with a significant shear layer thickness since $U(0) = 1$ only as $\theta \to 0$. In order to ensure that the jet center line velocity is exactly 1 several authors have considered the modified tanh velocity profile (4). For this velocity profile the first two terms in the series expansion of the velocity (16) are

$$U_0(s) = \frac{1}{2} \left[ 1 + \tanh \frac{s}{2} \right], \quad U_1(s) = \frac{s^2}{8} \sech^2 \frac{s}{2}.$$

Note that the $O(1)$ term $U_0(s)$ is identical to the previous tanh velocity profile. This implies that $W_0(s)$ is also unchanged and most of the previous results can be directly used for the modified tanh velocity profile (4).

The $O(1)$ equation from (17) is unaffected by the higher order terms of the velocity expansion so the $O(1)$ match is identical to the previous case. Since $P_0(s) = D_0$ is a constant the $O(\theta)$ equation in (17) is also unaffected by the new velocity profile. Differences with regard to the previous calculation start at the $O(\theta^2)$ equation, in the function $f_1(s)$ in the lhs. It is easily verified though that the asymptotic behavior of $P_1(s)$, $G_1(s)$ is the same as for the previous velocity profile so equations (35), (36) remain valid. We draw the conclusion that the modified velocity profile (4) leads only to $O(\theta^2)$ corrections in the eigenvalues. This was somewhat to be expected. The eigenvalue problem appears because of the necessity of matching interior and exterior solutions. This matching is effectively carried out in the boundary layer. Since in this zone (3) and (4) are identical to $O(\theta)$ the results are the same. These conclusions are also borne out by numerical calculations [10] in which results for the two tanh velocity profiles are practically identical.

4.2. INCOMPRESSIBLE, ISOTHERMAL JET WITH EXP VELOCITY DISTRIBUTION

We now consider velocity profiles (2) and (5). Both are of the form $U(s) = \exp \left[ -n(bs + 1) \right]$ with only a zero-order term in the $\theta$ power expansion. Let $y = \sqrt{n}(bs + 1)$. Integrating $W_0(s) = (U - c_0)^2$ we obtain

$$W_0(s) = c_0^2 y - c_0 \sqrt{\pi \text{erf}(y)} + \frac{\sqrt{\pi}}{8 \sqrt{2}}$$

The asymptotic behavior is

$$W_0(s) \sim \begin{cases} 
    c_0^2 y - c_0 \sqrt{\pi} + \frac{\pi}{8} & s \to \infty \\
    c_0^2 y - c_0 \sqrt{\pi} - \frac{\pi}{8} & s \to -\infty
\end{cases}$$
The $O(1)$ match leads to the same conclusions as before: $C_0 = 0$ and $A_I(\beta_0) = -BK_m(\beta_0)$. The $O(\theta)$ jet interior matching conditions are

$$D_0 = A_I(\beta_0)$$
$$C_0^2 \sqrt{n} \beta_0 A_I(\beta_0)$$
$$\left( c_0^2 \sqrt{n} + c_0 \sqrt{\pi} - \frac{\pi}{\sqrt{8}} \right) C_1 + D_1 = \beta_1 A_I(\beta_0)$$

and the jet exterior matching conditions are

$$D_0 = BK_m(\beta_0)$$
$$C_0^2 \sqrt{n} \beta_0 BK_m(\beta_0)$$
$$\left( c_0^2 \sqrt{n} - c_0 \sqrt{\pi} + \frac{\pi}{\sqrt{8}} \right) C_1 + D_1 = \beta_1 BK_m(\beta_0).$$

For this velocity profile we do not obtain the vortex sheet displacement continuity condition (14). This condition must hold in the limit $\theta \to 0$ so we conclude that exponential velocity distributions such as (2), (5) lead to errors of $O(\theta)$ in the eigenvalues and are inadequate descriptions of the jet mean flow. This conclusion has also been drawn from examination of numerical calculations (see the discussion of fig. 13 in [10]). Matching, such as above, permits us to determine the reason why exponential velocity profiles are inadequate.

5. THE PATCHING PROCEDURE FOR TANH VELOCITY PROFILE

We consider the patching procedure only for profile (3). Equations (25) truncated to $O(1)$ lead to

$$A_I(\beta_0(1 - n\theta)) = C_0 Wi(-n) + D_0$$
$$BK_m(\beta_0(1 + n\theta)) = C_0 Wi(n) + D_0$$
$$A\beta_0 A_I(\beta_0(1 - n\theta)) = \frac{1}{\theta} C_0 W_0(-n)$$
$$A\beta_0 BK_m(\beta_0(1 + n\theta)) = \frac{1}{\theta} C_0 W_0(n)$$
The determinant must be null for non-zero eigensolutions. The resulting dispersion relationship is

$$\theta \beta_0 \beta_m \left[ W(-n) - W(n) \right] I_m' K_n' = \beta_0 I_m K_n W_0(-n) - \beta_0 I_m' K_n' W_0(n)$$  \hspace{1cm} (37)

For $\theta \to 0$ ($n \to \infty$) we have $W_0(-n) \to (1 - c_0^2)^n$, $W_0(n) \to c_0^2$ so that for incompressible flow ($\beta_0 = \beta_0$) the vortex sheet dispersion relation (15) is obtained from (37) as a limiting case. Higher order dispersion relationships are also obtainable but are found to bring about insignificant corrections.

6. RESULTS AND COMPARISONS WITH THE STANDARD NUMERICAL PROCEDURE

The results of the previous sections have shown that in order to obtain $O(\theta)$ eigenvalues we need only consider the tanh velocity profile (3). Calculations are presented only for this case. Equations (31) and (35) are solved in order to obtain an $O(\theta)$ accurate eigenvalue estimate for the matching procedure. We consider the case of spatial instability of an isothermal, incompressible jet and seek complex eigenvalues $\alpha = \beta$ for real values of the frequency $\omega$. In order to obtain a real frequency $\omega = \omega_0 + \omega$, the $O(1)$ frequency $\omega_0$ must be chosen complex. Adequate values of $\omega_0$ are sought so that this condition may be fulfilled. Equation (37) is solved in order to present patching procedure results. The constant $n$ from (37) is determined by numerical experiment to be $3 \leq n \leq 4$. For comparison purposes the pressure equation is also solved numerically by a shooting procedure. The results are presented in figure 3.4 for the first two instability modes: the axially symmetric mode $m = 0$ and the first spinning mode $m = 1$ for $\theta = 0.02$. Continuous lines denote the matching results and broken lines denote the patching results.

![Graph showing Re \( \omega \) vs \( \omega \)](image)

Fig. 3 - Modes $m = 0, 1, \theta = 0.02$ (- matching conditions, -- patching conditions, * numerical solution)
Fig. 4 – Modes \( m = 0,1, \theta = 0.02 \) (– matching conditions, – – – patching conditions, • numerical solution)

Symbols are used for the results of the numerical computations. Agreement between the three procedures is excellent for low frequency values relative errors being of the order of 0.5%. At higher frequencies the matching procedure becomes less accurate (errors of \( \approx 5 - 10\% \)) especially for the imaginary part of the eigenvalues that determines the instability mode growth rate. The real part that determines the phase velocity is better predicted. The patching procedure gives better results in the high frequency range. Both matching and patching furnish accurate results only for the initial part of shear-layer growth up to \( \theta \approx 0.04 \). Computational costs are 10-20 times smaller than the numerical procedure.

7. CONCLUSIONS

An analytical approximation to the eigenvalues of the jet instability problem has been obtained by applying mathematical boundary layer theory. The principal results of the analysis are as follows:

- care should be taken in choosing analytical approximations to the mean-velocity. A class of mean velocity profiles based on the \( \exp \) function have been shown not to have correct asymptotic behavior;
- good approximations of the eigenvalues are obtained for the initial stages of the jet flow when \( 0.01 < \theta < 0.04 \). Computational costs are minimal;
- the analytical nature of the results is useful in studying the relationship between the local and global instability characteristics of jet flow. This is presented in detail in the third article in this series.

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