An approximated method for the solution of elliptic problems in thin domains: Application to nonlinear internal waves

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A B S T R A C T

Realistic numerical simulations of nonlinear internal waves (NLIWs) have been hampered by the need to use computationally expensive nonhydrostatic models. In this paper, we show that the solution to the elliptic problem arising from the incompressibility condition can be successfully approximated by a few terms (three at most) of an expansion in powers of the ratio (horizontal grid spacing)/(total depth). For an n dimensional problem, each term in the expansion is the sum of a function that satisfies a one-dimensional second-order ODE in the vertical direction plus, depending on the surface boundary condition, the solution to an n – 1 dimension elliptic problem, an evident saving over having to solve the original n-dimensional elliptic problem. This approximation provides the physically correct amount of dispersion necessary to counteract the nonlinear steepening tendency of NLIWs. Experiments with different types of NLIWs validate the approach. Unlike other methods, no \textit{ad hoc} artificial dispersion needs to be introduced.

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1. Introduction

Nonlinear internal waves (NLIWs) have emerged in the past 30 years as a prominent feature of many shelf and coastal areas around the world. For a comprehensive catalog of observations, the reader is referred to Jackson (2004). In addition to being an interesting problem in itself, NLIWs impact several areas of coastal oceanography through enhanced mixing and transport (MacKinnon and Gregg, 2003; Leichter et al., 2003; Moun et al., 2003), biological oceanography by redistributing plankton (Pineda, 1999; Helfrich and Pineda, 2003; Scotti and Pineda, 2007), and geological oceanography by suspending and transporting sediments (Bogucki et al., 1997; Butman et al., 2006).

For this reason, much research has been devoted to modeling NLIWs (Helfrich and Melville, 2006). Though reliable quantitative observations of NLIWs were available since the late 1960s (Ziegenbein, 1969; Halpern, 1971) they attracted serious consideration in the early 1980s (Osborne and Burch, 1980), when it was recognized that weakly nonlinear wave theory could be used to frame the problem (Benney, 1966; Liu and Benney, 1981). The assumption was that, properly normalized, the amplification in the early 1980s (Osborne and Burch, 1980) could be treated as small parameters in a series expansion. To the lowest nontrivial order this leads to the Korteweg–de Vries (KdV) equation (Korteweg and de Vries, 1895) for the amplitude of the waves.\textsuperscript{1} The principal insight of KdV theory is that the steepening tendency of nonlinearity is balanced by the dispersive nature of the medium in which the waves propagate. This allows finite amplitude waves of special shape to propagate without distortion. While attractive for its elegance and simplicity, KdV theory suffers from many shortcomings, which limit its usefulness as a predictive tool. It is well known that NLIWs in the ocean are often highly nonlinear, quite steep and have trapped cores (Stanton and Ostrovsky, 1998; Klymak and Moun, 2003; Scotti and Pineda, 2004); during generation and shoaling, topography couples modes, whereas KdV neglects mode-mode interaction and cannot handle steep topography; over long propagation times, rotation is important (Helfrich, 2007); three-dimensional effects can only be incorporated assuming weak dependence on the direction normal to the propagation; dissipation and instabilities require \textit{ad hoc} treatment. While some of these concerns can be addressed within the KdV framework (Grimshaw and Smyth, 1986; Grimshaw et al., 1999; Smyth and Holloway, 1988; Holloway et al., 1997; Holloway et al., 1999), the fundamental limitations of the weakly nonlinear framework cannot be escaped. For this reason, newer models have been intro-

\textsuperscript{1} In the two-layer approximation, the amplitude is the displacement of the interface separating the two layers; for a continuous stratification, waves are projected onto the normal modes of the linear problem, and KdV describes the evolution of the amplitude of a particular mode, in a frame of reference traveling with the linear phase speed of the mode.
duced, which typically remove the weakly-nonlinear constraint. The prototype is the (Choi and Camassa (1999)) model. This is a two-layer model, fully nonlinear but still dependent on the steepness of the waves to be small. It can be generalized to multiple layers, and can handle smooth topography, but it is not free from issues. For example, any amount of shear between layers, however small, will trigger Kelvin–Helmholtz instabilities that need to be filtered out. At the opposite end of the complexity scale we have the well established general circulation models (GCMs). Robust, well documented, with an extensive set of tools to address biological and geophysical problems, they have been used to address a wide range of problems. From an operational point of view, they would be the tool of choice to study NLIWs. Unfortunately, they have traditionally taken advantage of the hydrostatic approximation to reduce the computational load, and thus cannot provide the dispersion needed to counteract nonlinearity. Newer models, such as SUNTANS (Fringer et al., 2006) or the MITgcm model (Marshall et al., 1997) have been developed that do not make the hydrostatic approximation, and have been used to study NLIWs in realistic settings. However, despite the increase in computational power in the last decade, running these models for realistic problems is still extremely expensive. Not surprisingly, for these runs a significant fraction of the cost (as high as 60%, Fringer, personal comm.) is taken up by the solution of the three-dimensional elliptic problem associated to the incompressibility condition. This limits severely the resolution that can be achieved, and thus casts reasonable doubts on the predicted characteristics of NLIWs.

In this paper, we show that it is possible to relax the nonhydrostatic constraint so that a (suitably modified) nonhydrostatic model such as SUNTANS could be run to realistically simulate NLIWs without incurring the full cost of the nonhydrostatic case. The main objective is to introduce the appropriate amount dispersion in a controlled way, with tools that are available to a nonhydrostatic ocean model, while keeping the numerical overhead at a minimum. The method is based on a perturbative approach to the elliptic problem, inspired by how the theoretical models are derived. The crucial insight however is to recognize that the critical length scale upon which to base the expansion is not the physical length scales of NLIWs, but the numerical horizontal length scale of the grid. This idea may sound foreign to a mind accustomed to consider theoretical models as continuum objects which may eventually be solved numerically (if everything else fails). However, it is the natural way to approach the problem if we subscribe to the view that a GCM is a discrete numerical tool which tries to model a continuum system (the ocean).

Of course, it would be desirable to implement a similar strategy using a hydrostatic model as a starting point. As it will become clear during the foregoing discussion, this can be done rather easily if the hydrostatic model makes the rigid-lid approximation. If, however, the model uses a free-surface, application of the method described in this paper is not straightforward. Appendix A discusses some of the issues at stake. However, for the free-surface case none of them is satisfactory; a complete analysis is beyond the scope of this paper.

2. Analysis

The hydrostatic approximation is often introduced on dynamical grounds (see, e.g. Haidvogel and Beckmann, 1999, p. 21) whereas in fact, as will be shown below, it follows from a combination of kinematic and geometric constraints. Flows that occur on a horizontal scale $L$ much larger than the local depth $H$ are hydrostatic even in a microgravity environment. Conversely, it is wrong to treat as hydrostatic flows with scales $L \approx H$. The issue with NLIWs is that their horizontal length scale is typically $O(H)$. Fig. 1 shows the width and maximum isopycnal slope of steady solitary waves generated with the Dubreil–Jacotin–Long (DJL) equation (Dubreil–Jacotin, 1937; Long, 1953) vs. wave amplitude $\eta_0$. To generate a wave, a solution of the DJL equation is obtained for a given total available potential energy (APE) (Scotti et al., 2006) using the technique described in Lamb and Wan (1998). For small APE values, the width $L_w$ of the wave follows the KdV scaling $L_w = O(\eta_0^{1/3})$. For the particular stratification considered here (standard hyperbolic tangent pycnocline), the width bottoms out at $\eta_0/H \sim 2$ (i.e. when amplitude matches the depth of the pycnocline), after which it begins to grow, as the solution approaches the conjugate state (Lamb and Wan, 1998). As the APE increases, the amplitude $\eta_0$ saturates as well, as can be seen from the flattening of the maximum slope curve. In other words, the depth of the pycnocline limits the amplitude of the waves. From a numerical point of view, these results suggest that an adequate horizontal resolution is $O(H/10)$, which is confirmed by numerical experiments (Scotti et al., 2007). On these scales, the hydrostatic approximation breaks down, and seems to imply that a correct simulation requires the use of a nonhydrostatic code. In the following, we show that this approach is overly conservative. It is possible to relax the nonhydrostatic condition while still having the correct dispersive behavior on the scales relevant to NLIWs propagation.

2.1. Solution method

Within an Euler solver, the incompressibility condition slaves the pressure to the instantaneous velocity and buoyancy field via an elliptic operator, the price paid for having filtered out the fast acoustical modes. Numerically, this means that we have to deal at some point with a Poisson problem, which for a generic curvilinear coordinate system, takes the form (Aris, 1989, p. 169–170)

$$
\partial_i (g^i \partial_j \phi) = \partial_i (Ju^j),
$$

(1)

The unknown $\phi$ can be the pressure or the potential in a projection scheme, but could also be the geopotential in a pressure coordinates

\footnote{This observation is of course not new, for example it is briefly hinted in Pedlosky (1986, p. 61).}
scheme. For simplicity, we have written the r.h.s. as the divergence of a vector \( \mathbf{w} \) (whose contravariant components are \( w^i \)), but it could also be a generic scalar function. Finally, \( \partial_i \equiv \partial/\partial x^i \), \( g^i_0 \) is the contravariant metric tensor of the coordinate system, and \( J \equiv 1/\text{Det}(g^0) \).

Note that, by suitable redefining \( g^i_0 \), it is possible to consider more complex cases of non constant coefficients elliptic operators. Unless otherwise stated, we use the Einstein notation on repeated indexes when the sum runs from 1 to 3, while we use explicit summation when the sum runs from 1 to 2.3

Because of the linearity of the problem, the elliptic operator does not posses an “intrinsic” length scale. Thus, the relevant horizontal length scales of the problem are the ones introduced by the topography via the metric tensor (if present) \( L_p \), by the physics \( L_p \) (in the r.h.s.) and by the numerical discretization \( \Delta x \sim \Delta y \). Of the three, the last one is necessarily the smallest. Moreover, it is always possible to rescale the potential \( \phi \) such that \( \phi = O(1) \). Let \( H \) be a depth scale (such as the maximum depth in the domain). We want to exploit the fact that the processes we are interested in have horizontal scales comparable or larger than the local depth. Define the grid leptic ratio \( \lambda \equiv \Delta x/\Delta y \). For a smooth function \( f \), the upper limit of the magnitude of the horizontal derivatives is \( O(\Delta y/\Delta x) \), whereas the lower limit of the magnitude of the vertical derivative is \( O(\Delta y/\Delta x) \).

The key assumption is that (no summation over repeated indices)

\[
\partial_i g^{ij} \partial_j \phi = \begin{cases} O(\lambda^2) & \text{if } i \neq j \neq 3, \\ O(1) & \text{if } i = j = 3, \end{cases}
\]

where \( \lambda \equiv \lambda^{-1} \). The idea is to turn the fact that the operator is badly conditioned to our advantage. To do this, we seek a formal expansion of the solution in powers of the inverse square of the grid leptic ratio

\[
\phi = \phi_0 + \lambda^2 \phi_1 + \lambda^4 \phi_2 + \cdots
\]

While \textit{a priori} it may not be obviously important why this is the appropriate scaling, an \textit{a posteriori} justification will be given below when discussing the convergence of the scheme. To the lowest order we have

\[
\partial_i (g^{ij} \partial_j \phi_0) = \partial_i (J u^i) .
\]

If Dirichlet conditions are specified on either lower or upper boundary, Eq. (4) completely specifies \( \phi_0 \). The next order correction is found by replacing \( u^i \) with \( u^i - g^{ij} \partial_j \phi_0 \) and \( \phi_0 \) with \( \phi_1 \) in Eq. (4) and so on.

If instead along both the lower and upper boundary \( g^{ij} \partial_j \phi_0 \) is specified (Neumann conditions), additional steps need to be taken, since Eq. (4) specifies \( \phi_0 \) only up to an additive function \( \phi_0^a(x, z) \). That is \( \phi_0 = \phi_0^\text{a} + \phi_0^\text{b} \) with \( \phi_0^\text{b} \) a solution of Eq. (4). To specify \( \phi_0^\text{b} \) we need to consider the problem at order \( \lambda^2 \):

\[
\sum_{i,j=1}^2 \partial_i (g^{ij} \partial_j \phi_0^\text{b}) + \partial_3 ( \sum_{i,j=1}^2 g^{ij} \partial_j \phi_0^\text{b} + g^{3j} \partial_j \phi_1 ) = \partial_3 (J u^3 - g^{ij} \partial_j \phi_0^\text{a}) ,
\]

which now contains two unknowns: \( \phi_0^\text{b} \) and \( \phi_1 \). However, since \( \phi_0^\text{b} \) does not depend on \( x^3 \), we can integrate right and left hand side of Eq. (5) along \( x^3 \) to obtain the following two-dimensional elliptic problem for \( \phi_0^\text{b} \):

\[
\sum_{i,j=1}^2 \partial_i (g^{ij} \partial_j \phi_0^\text{b}) = \sum_{i,j=1}^2 \left\{ \partial_3 (\partial_3 (J u^3 - g^{ij} \partial_j \phi_0^\text{a})) \right\} = 0 ,
\]

and obtain the second term in the r.h.s. from the Neumann boundary conditions. As lateral boundary conditions we use the vertically integrated boundary conditions, the reminder being satisfied by an appropriate extrapolation of \( \phi_0^\text{b} \) along the lateral boundaries. Once Eq. (5) is solved, \( \phi_0^\text{a} \) is fully determined (up to an inconsequential additive constant).

To replace \( u^3 \) with \( u^3 - g^{ij} \partial_j \phi_0^\text{a} \) we can now repeat the algorithm to determine \( \phi_1 = \phi_1^\text{a} + \phi_1^\text{b} \) and so on.

We have thus derived an iterative scheme to solve a generic elliptic problem subject to the conditions laid out in Eq. (2). No assumptions were made on the nature of the unknown, the r.h.s. or the boundary conditions. With regard to the latter, Dirichlet conditions require only trivial quadrature along the \( x^3 \) direction, whereas Neumann conditions at both \( x^3_{\text{lower}} \) and \( x^3_{\text{upper}} \) require an additional horizontal elliptic problem at each step. Finally, each term in the expansion is obtained solving the same set of problems.

2.2. Convergence

The procedure outlined above, while formally correct, is not guaranteed to converge to the actual solution. In fact, the series in Eq. (3) converges only if the grid leptic ratio is greater than a critical value \( \lambda_c = O(1) \).

To show why this occurs, we consider a simple constant-depth bidimensional channel geometry with homogenous Neumann conditions at top and bottom \( \partial \phi / \partial z = 0 \). For this simple geometry, the problem can be numerically diagonalized by taking a cosine transform in both directions. Thus, it is sufficient to consider the following problem:

\[
\begin{align*}
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} &= \cos(kx) \cos(qz) \\
\end{align*}
\]

on the domain \( 0 < x \leq L, 0 < z \leq H \), and \( k = n \pi/L, q = m \pi/H \) for arbitrary \( n \) and \( m \). On a fixed grid, using the standard 5-point stencil for the Laplacian, the exact numerical solution on a grid discretized with \( N_x \) points in the horizontal direction and \( N_z \) points in the vertical is

\[
\phi = \frac{\cos(kx) \cos(qz)}{k + q} .
\]

where \( k \equiv k_0 (2N_z^2 (1 - \cos(kL/N_x))/L^2, q \equiv q_0 = 2N_z^2 (1 - \cos(qH/N_z)) \) are the numerical eigenvalues of the operator. The solution method outlined in the previous section gives the solution as a McLaurin series in \( k \). However, from Eq. (8) above, we see that the radius of convergence of the McLaurin series in \( k \) for \( \phi \) is determined by the location of the poles in the complex plane of \( f_0(z) = (z + q_0)^{-1} \) for a given \( q \). Convergence thus occurs only if \( |z| < |q| \). The worst-case scenario happens when \( q = \pi/H \) and \( k = N \pi/L \). Thus, convergence occurs when \( \Delta x/\Delta y = \lambda > 2 \pi + O(N_z^{-2}) \approx \lambda_c \). Other discretizations may have different radii, but always such that \( \lambda_c = O(1) \). Thus, if the grid leptic ratio is smaller than a critical value, even if the topographic and physical scales of the problem are such that \( \text{min}(L_x, L_y) \gg H \), the iteration will fail because of its essential instability on the supercritical modes. Incidentally, this analysis vindicates the choice of the inverse leptic ratio as scaling parameter in the derivation of the expansion.

This argument can be extended to the arbitrary case along the following line. Under general assumptions, the eigenvalues of the Laplacian form an unbounded set. Hence, the extension of the Green’s function to the complex plane defines a bounded function,
which thus must have poles in the complex plane. As in the case considered above, the location of these poles determines the radius of convergence of the method.

Superficially, this property would seem to limit the applicability to the present method to rather "coarse" grids. In particular, $L_p$ for NLIWs is $O(H)$ (Grimshaw, 2001; Grimshaw, 1997). Considerations of numerical accuracy require $\Delta x = O(L_p/10)$, thus apparently placing NLIWs out of reach. This would indeed be the case if the goal is to calculate the potential (physically related to the pressure) $\phi$ to a predetermined accuracy. However, if the goal is instead to introduce the appropriate amount of dispersion on the scales relevant to NLIWs, convergence by itself is not important. Rather, it is the behavior of the truncated series that needs to be investigated.

2.3. The expansion arrested to the $M$th order

To understand the physical consequences of arresting the expansion to order $M$, consider the following linearized inviscid bidimensional internal wave propagation problem in a channel of fixed height $H$

$$
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial p}{\partial x}, \\
\frac{\partial w}{\partial t} &= \frac{\partial p}{\partial z} - b, \\
\frac{\partial b}{\partial t} &= wN^2, \\
\nabla \cdot u &= 0,
\end{align*}
$$

with Neumann condition at the upper and lower boundaries $\partial p/\partial z = -b$, whose solution we denote as $p = P(b)$. Dynamically, the above system is equivalent to

$$
\frac{dq}{dt} = Aq, \quad q = (u, b),
$$

and

$$
A = \begin{pmatrix} 0 & -\frac{\partial p}{\partial x} \\ -N^2 \frac{\partial}{\partial z} \int_0^z \frac{\partial p}{\partial x} & 0 \end{pmatrix},
$$

since $p$ and $w$ are slaved to $u$ and $b$ (the square brackets are to be filled with field the operator is acting upon). If Eq. (10) is solved exactly, the eigenvalues $\omega$ of $A$ are given by

$$
\frac{\partial^2 \psi}{\partial x^2} + \frac{\omega^2 + N^2}{\omega^2} \frac{\partial^2 \psi}{\partial z^2} = 0, \quad \psi = \frac{\omega^2}{\omega^2 + N^2} \frac{\partial p}{\partial z},
$$

with homogeneous Dirichlet/Neumann boundary conditions on $\psi$ inherited from $p$ (for a derivation, see Appendix B). If instead the approximation outlined in the preceding section is used retaining the first $M$ terms, we derive the following eigenvalue problem

$$
\left( \begin{array}{cc} \omega^2 + N^2 & \frac{M}{n=0} \left( -\frac{\partial^2}{\partial z^2} \right)^n \end{array} \right) b = 0,
$$

where

$$
\int |b| = \int_0^T ds \int_0^T dq b(q, x) - \frac{Z}{H} \int_0^H ds \int_0^T dq b(q, x),
$$

is a compact operator (a consequence of the Arzelà-Ascoli theorem, Rudin, 1987, p. 245), and $b$ has the same mixed homogeneous Dirichlet/Neumann boundary conditions as $\psi$. Note that the approximated pressure is given by

$$
P(b) = -\frac{1}{H} \sum_{n=0}^M \left( -\frac{\partial^2}{\partial z^2} \right)^n b.
$$

In particular, at the lowest order ($M = 0$) Eq. (16) becomes

$$
P(b) = -\int_0^T ds b - \frac{1}{H} \int_0^H dz \int_0^T ds b,
$$

and our approximation recovers the hydrostatic limit. For this simple case (linearized and constant depth), the solution to the horizontal elliptic problem, which gives the so-called "lid pressure", is obtained trivially. This is not the case when nonlinear terms and/or a non constant depth are included in the problem. Applying the operator $\sum_{n=0}^M \left( -\frac{\partial^2}{\partial z^2} \right)^n$ to Eq. (13) and letting $\psi = N^2 \psi$ we obtain

$$
\left( \begin{array}{cc} \omega^2 + N^2 & \frac{M}{n=0} \left( -\frac{\partial^2}{\partial z^2} \right)^n \end{array} \right) \psi
$$

(18)

Letting $k^2$ be the (numerical) eigenvalue of the operator $-\partial^2/\partial z^2$, Eq. (18) shows that the eigenvalues of $A$ when the elliptic problem is solved exactly agree with the eigenvalues obtained from an approximated solution up to $O(k^2M^{-1})$. Further insight comes from the characterization of the eigenvalues defined by Eq. (14) as a function of $M$. For convenience, we carry on the analysis in terms of the phase speed $c^2 = -\omega^2/k^2$.

2.4. Small $k$ behavior

2.4.1. $M = 0$

In this case, Eq. (14) is equivalent to

$$
\frac{d^2 \psi}{dz^2} + N^2 \frac{\partial}{\partial z} \psi = 0,
$$

(19)

which is the Taylor–Goldstein equation of the shear-free hydrostatic problem (Kundu and Cohen, 2008), p. 500. This shows that the hydrostatic behavior is in fact a consequence of the flattened aspect ratio of the domain.

2.4.2. $M > 0$

A simple asymptotic expansion shows that $c_{2,n}^2 = c_{2,0}^2(1 - \alpha_n k^2) + O(k^4)$, where $c_{2,0}^2$ are the eigenvalues of Eq. (19),

$$
\alpha_n = \frac{\int_0^H \psi_0^2 dz}{\int_0^H (\psi_0^2/\partial z) dz} dz,
$$

(20)

and $\psi_0$ are the eigenvectors of the hydrostatic Taylor–Goldstein equation. To $O(k^4)$, this dispersion relationship is identical to the one obtained linearizing the KdV equation describing weakly nonlinear internal waves in a continuously stratified medium (see e.g., Benney, 1966; Liu and Benney, 1981).

2.5. Large $k$ behavior

The short wavelength behavior depends critically on the parity of $M$. For large $k$, let the eigenvalues be $c_{2,n}^2 = k^2 M_n + O(k^{-1})$. To leading order, $c_{2,0,n}$ are the eigenvalues of

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For a discussion of the appropriate boundary conditions under free-surface conditions, see Appendix A.
\( (c^2 + N^2) \psi = 0, \quad \psi(0) = \psi(H) = 0. \) \hspace{1cm} (21)

If \( \psi \) is an eigenfunction, with eigenvalue \( c \), then it is trivial to show that
\[
c^2 = (-1)^{M} \int_{0}^{H} d (N^2) \left( \frac{\psi}{dz} \right)^2 dz,
\]
\hspace{1cm} (22)

with \( \psi \equiv \psi + i \bar{\psi} \). It follows that asymptotically, the sign of the eigenvalues is given by \( (-1)^{M} \), provided \( N^2 > 0 \). The somewhat surprising consequence is that for \( M \) odd, shortwavelength waves are unstable, since to a negative \( c^2 \) corresponds a real eigenvalue \( \omega > 0 \) of \( A \) in Eq. (11).

2.6. Finite \( k \) behavior

So far we have established that as \( k \to 0 \) \( c^2 \) is always positive, while as \( k \to \infty \) the sign depends on the parity of \( M \). To determine the sign of \( c^2 \) for intermediate wavelength, we need to examine if and where \( c^2 \) crosses the real axis.\(^7\) For this to occur, the eigenfunction \( b \) must satisfy
\[
\sum_{n=0}^{M} k^2 n^2 \psi_{n} \xi_{n} = Q h = 0.
\]
\hspace{1cm} (23)

Since \( \sin(m \pi z)/H, m = 1, \ldots \) diagonalizes \( Q \), this occurs if and only if solutions to
\[
\sum_{n=0}^{M} (-1)^{n} \left( \frac{kH}{m \pi} \right)^{2n} = 0
\]
\hspace{1cm} (24)
can be found for \( k > 0 \). For this to happen \( M \) must be odd, in which case the only zeros on the real axis are \( \frac{m \pi}{h} = \pm 1 \).

To summarize, if \( M \) is odd and \( k > \pi/H \) (that is, if \( k > \lambda_{c} \)) Eq. (11) is an ill posed problem. If, on the contrary, \( M \) is even, or \( k > \lambda_{c} \) Eq. (11) is a well posed problem. Moreover, the dispersion relationship of the problem approximated to order \( M \) agrees with the exact dispersion relationship with an error which is \( O(k^{2(M-1)}) \) at long wavelength. Fig. 2 shows the dependence of \( c^2 \) on wavenumber for \( M = 0,1 \) and 2. The eigenvalues were calculated numerically assuming for \( N^2(z) = N_{0}^2 \text{sech}^2((0.3H - z)/0.1H)z \in [-H, 0] \), using 26 levels in the vertical. The phase speed is normalized with the exact phase speed, to highlight the departure from the correct dispersion relationship with an error which is \( O(k^{2(M-1)}) \) at long wavenumber. The vertical line at \( \pi \) marks the boundary between wavenumbers that are stable for all \( M \) from wavenumbers that are stable only when \( M \) is even. The critical value of the grid leptict ratio for the onset of unstable modes depend on the numerical discretization. For a standard second order discretization, unstable wavenumbers exist when \( k < \pi/2 \). For a spectral discretization, \( k < \lambda_{c} \). For unstructured grids, the value must be determined experimentally, but it is always \( O(1) \).

3. Application to NLIWs

The analysis presented above shows that the truncated expansion in Eq. (3), when applied to linear internal waves, provides the right amount of dispersion for wavenumbers \( k \approx \pi/H \). To test how well the method works with nonlinear waves, as well as to explore numerical issues, we consider the fully nonlinear Euler equations for a fluid of variable density (under Boussinesq approximation) in a two-dimensional channel, which are solved numerically using a standard operator splitting scheme. We briefly describe how time is marched forward, because the particular nature of the dispersion relationship in the approximated case requires proper numerical treatment.

3.1. Time advancement

If \( \mathbf{s}' = (u', w', b') \) represents the state of the system at time \( t_{n} \), first an intermediate solution is generated
\[
\mathbf{s}^{n+1} = \mathbf{s}^{n} + \Delta t \mathbf{A} \mathbf{s}^{n},
\]
\hspace{1cm} (25)

by considering only the nonlinear terms. \( \Delta t \) is controlled by the usual Courant–Friedrichs–Lewy condition (Durran, 1999). To the intermediate solution we then add the linear part and extract its solenoidal component to obtain \( \mathbf{s}^{n+1} \) using an Adams–Moulton method for time advancement (Canuto et al., 1987, p. 104). That is,
\[
t^{n+1/2} = u^{n},
\]
\[
w^{n+1/2} = \nabla \cdot \mathbf{u}^{n+1/2} + (1 - \theta) b^{n},
\]
\[
\theta b^{n+1} = b^{n} + \Delta t N^{2} (\theta w^{n+1} + (1 - \theta) w^{n}),
\]
\[
t^{n+1} = u^{n+1/2} + \nabla \phi/\partial x,
\]
\[
w^{n+1} = \nabla \cdot \mathbf{u}^{n+1/2} + \nabla \phi/\partial z,
\]
\[
\nabla \cdot \mathbf{u}^{n+1} = 0.
\]
\hspace{1cm} (26)

Eq. (26) is the discretized version of Eq. (9). The parameter \( \theta \in \{0, 1/2, 1/2 + z \Delta t, 1\} \) determines the time advancement scheme. The potential \( \phi \) required by the projection step is determined by
\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial z} - 1 + \theta^2 \Delta t^2 N_{0}^2 \frac{\partial}{\partial z} \right) \phi = \text{r.h.s.},
\]
\hspace{1cm} (27)

which can be solved either exactly, or by means of the approxima-
\( \lambda > \lambda_c \), the eigenvalues \( \omega = \pm i c|k| \) lie on the imaginary axis (Section 2.5)

\[
|\omega \Delta t| \leq \frac{\gamma c(0)\Delta t}{\Delta x} \leq \frac{\gamma c(0)}{\max(u, w)} \frac{\max(u, w)\Delta t}{\Delta x},
\]

and \( \gamma \) is an \( O(1) \) constant. Given that \( c(0)/\max(u, w) \) is bounded, stability of this step is controlled by the Courant–Friedrichs–Lewy condition, provided the time discretization can handle purely imaginary eigenvalues. In practice, we found that a Crank–Nicholson scheme gives satisfactory results. If, on the contrary, the expansion is arrested at the second order and \( \lambda < \lambda_c \),

\[
\max(|\omega \Delta t|) \sim \frac{\Delta t}{\Delta x^2},
\]

and care has to be exercised in how time is discretized. In practice, we found advisable to use an implicit scheme which damps the high-wavenumber components, either Euler–Backward \( \theta = 1 \) or the so-called \( \theta \)-method, \( \theta = 1/2 + x\Delta t \), with \( x \) a small positive constant. The latter has the advantage of being formally second-order accurate in time. The rationale for numerically damping the high wavenumbers lies in the fact that (i) they are needed only to ensure accuracy in calculating the nonlinear fluxes and (ii) they move with the “wrong” phase speed.

### 3.2. Propagation of NIWVs

In these experiments, the stratification in the 100-m deep channel is given by

\[
N^2 = \frac{N_0^2}{\cosh^2((z_0 - z)/\delta)}
\]

\( z_0 = 20 \text{ m}, \delta = 10 \text{ m}, -100 \text{ m} \leq z \leq 0 \text{ m}, N_0 = 0.04 \text{ s}^{-1}. \]

The channel extends 13 km in the horizontal. At \( t = 0 \), a mode-1 wave is generated from the streamfunction \( \psi = c\eta(x, z) \), where \( \eta \) is the displacement field and \( c \) the nonlinear wave speed. The buoyancy anomaly is similarly defined as \( b = b(z - \eta) - b(z) \), with \( b \) being the stratification far upstream. The displacement field is written as \( \eta = \eta_f(x)\phi(z) \), with \( \phi \) the gravest eigenvector of Eq. (19), and \( f(x) \) the waveform. The waves are followed for 4 h. In all experiments, the vertical axis is discretized with 26 equispaced levels. The horizontal discretization is varied to give a grid leptic ratio in the range \( 25 < \lambda < 2 \).

In one set of experiments the initial shape of the wave is a soliton \( \text{sech}^2 \) with amplitude \( \eta_0 = 5 \text{ m} \), which approximates well the solution of the DJL equation (Fig. 3). Accordingly, this wave form should propagate without changing shape. In a second set of experiments, the initial shape is a square well (Fig. 3) of amplitude \( \eta_0 = 8 \text{ m} \). In the latter case, the wave is expected to evolve into an undular bore (sometimes called a solibore; Apel, 2003) under the combined effect of nonlinearity and dispersion. The aim of the latter experiment is to test the effectiveness of the expansion in generating NIWVs with the correct shape and frequency relative to the exact treatment of the elliptic problem. Note that the emphasis here is how well the waveform predicted with the truncated series matches the one obtained with the exact numerical solution of Eq. (27), not how well the solutions match the analytic solution. Finally, note that for this type of stratification, the maximal amplitude, as given by the DJL equation, is the depth of the pycnocline. The examples considered here are thus 25–50% of the maximal amplitude, and as such quite nonlinear.

In both cases, the amplitude of the initial disturbance is set large enough to make the contribution of the nonlinear terms significant, as shown by the fact that once sufficient resolution and dispersion is allowed, the soliton propagates without change in shape. For \( M = 0 \) (hydrostatic case), the only significant source of dispersion occurs in the numerical treatment of the nonlinear advection term. For the present calculations, we use a second order Gudunov scheme whose leading order error is dispersive. This introduces a term proportional to \( \Delta k^2 \) in the dispersion relationship whose magnitude is \( O(\Delta k^2 U_{zc}) \), and thus decreases as the leptic ratio is reduced, so that when \( M > 0 \) (and \( j \) is small) the main source of dispersion comes from the terms of the expansion, which have the correct asymptotic behavior at long and intermediate wavelengths. In the following, \( M = \infty \) will denote the exact solution of the elliptic problem in Eq. (27).

#### 3.2.1. Solitons

No differences between the \( M = 0 \) and \( M > 0 \) cases are observed when \( \lambda \gg 1 \). At \( \lambda = 1 \) the \( M = 1 \) and \( M = \infty \) solutions have essentially the same shape, which remain symmetric relative to the mid-point of the wave (Fig. 4). The hydrostatic solution (\( M = 0 \))

![Fig. 3. Initial waveforms for the square wave and the soliton. Note that since the pycnocline is close the surface, in the simulation they are taken as waves of depression.](image)

![Fig. 4. Slope of the waveform for a soliton that has been propagating for 4 h. The amplitude of the solitary wave is 4.5 m and the grid leptic ratio is \( \lambda = 1 \). The wave is plotted in a frame of reference moving with the wave. The shape of the wave in the \( M = 1 \) case is virtually indistinguishable from the exact case, and symmetric relative to its midpoint. Not so for the \( M = 0 \) case. The wave is steeper along the leading edge, and is also more compressed in the horizontal directions. Both effects are due to lack of adequate dispersion.](image)
however begins to develop a fore-aft asymmetry, with the leading edge steeper than the trailing edge, and with an overall “compression” of the horizontal scale, both consequences of lack of proper dispersion. Lowering the leptic ratio past the critical value further degrades the hydrostatic solution. In this case, a second wave begins to appear behind the first, driven by the artificial numerical dispersion. Unlike the soliton case considered below, the solution never achieves a steady state (in a frame of reference moving with the crest). Rather, the number of oscillations grows with time, being rank ordered, with the largest near the leading edge, and tapering off at the trailing edge. In the strongly nonlinear case, the picture is similar, but often only a few waves form (contrast, e.g., figure 12 and 13 in Scotti et al., 2007). The wavenumbers supported by a coarse grid ($\lambda = 2$) are not dispersive, so that in both $M = 0$ and $M = \infty$ cases the only dispersion to act is the numerical one, resulting in similar waveforms evolving from the initial square well (Fig. 6). As the grid is refined past the critical point, dispersive wavenumbers are added. When $\lambda = 0.5$ the $M = 2$ and $M = \infty$ solutions show smaller waves, with shorter wavelengths and with the wrong phase.

4. Discussion

We have proposed an algorithm that captures the dispersive properties of the nonhydrostatic Euler equations while avoiding the stiff penalty represented by the solution of the elliptic problem that arises from the incompressibility condition. The method is based on an asymptotic expansion of the elliptic operator arising from the incompressibility condition. We have tested the method on two-dimensional NLIWs. For grids with a grid leptic ratio $\Delta x/H$ greater than 2 there is no advantage in going beyond the $0^{\text{th}}$ order, which we have shown to recover the standard hydrostatic approximation. As the leptic ratio is reduced, dispersive wavenumbers are supported by the grid, and we have shown that adding the first-order term to the expansion introduces the correct amount of dispersion. At this order, to $O(k^3)$ the dispersion relationship that we obtain is identical to the one found in the KdV equation (Benney, 1966; Liu and Benney, 1981). As the leptic ratio is reduced past a $O(1)$ critical value (the exact value depends on the specifics of the spatial discretization), the second term in the expansion needs to be added to maintain stability. This result is somewhat surprising, given that KdV or the Choi and Camassa model (both of which are based on first-order expansions) do not suffer from this problem. For the KdV model, the reason lies on the fact that in deriving the model, a single branch of the
dispersion relationship is selected a priori. This prevents the occurrence of unstable modes, but also limits its applicability to unidimensional wave propagation. In the Choi and Camassa the reason is more subtle. In our approach, incompressibility is strictly enforced, and the pressure (or potential) is diagnosed from the instantaneous value of the state vector $s$. In Choi and Camassa, a regularization is introduced whereby the pressure is a function of both $s$ and $ds/dr$. This artificial “compressibility” stabilizes the high-frequency wavenumbers. Note however that the Choi and Camassa model is always unstable to shear instabilities, unless filtering is applied to suppress them.

Experimenting with different NLIWs we have found that it is never necessary to go beyond the second order in the expansion. This is due to the fact that the spectrum of NLIWs decays exponentially with an $e$-folding scale which is of the order of the local depth. The expansion provides the correct phase speed on these wavelengths. High wavenumbers are only necessary to enforce a reasonable accuracy on the calculation of the derivatives and we have found expedient, at small values of $\lambda$ to filter out numerically (using Euler Backward) the high-frequency components. As a rule of thumb, a leptic ratio of $0.3$ should give consistent results but some experimentation may be needed for a given problem to verify robustness.

We have tested these ideas on two-dimensional problems, but clearly the main advantages are reaped when dealing with three-dimensional problems. Each term in the expansion requires the solution of a 1D second-order ODE at each point in the horizontal direction, plus a single two-dimensional elliptic problem to determine the “lid” pressure if the rigid lid assumption is used. Assuming a structured grid with $N_x \times N_y \times N_z$ points, the first part requires $O(N_x N_y N_z)$ operations. To achieve convergence for the 2D problem a multigrid solve would cost $O(N_x N_y \log(N_x N_y))$, given that the two-dimensional problem is well balanced, as opposed to the three-dimensional one, which is dominated by the vertical derivatives, the ratio of which $H/h > 1$ influences the number of multigrid iterations needed for convergence, in accordance with the estimate $N_x N_y N_z \log(N_x N_y N_z/H/h)$. Even when three terms in time are needed, the total cost is only a small fraction of the cost of solving the three-dimensional elliptic problem. For a typical case with $N_x = N_y = 256$, $N_z = 32$, $H/h = 10$ the three-dimensional procedure requires $3–6$ times more work. From an implementation point of view, there should be no problem in coding the algorithm within a nonhydrostatic code. “Upgrading” a hydrostatic code, especially a free-surface one, is going to be more challenging, and we discuss the issues at stake in Appendix A.

The focus of this paper was on Nonlinear Internal Waves in a Boussinesq fluid. However, the method relies only on the skewed geometry of the domain, and can be used whenever this condition is satisfied. For example, it can be applied to the less restrictive anelastic approximation

$$\text{div}(\mathbf{u}) = 0,$$  \hspace{1cm} (31)

given that the ensuing nonconstant coefficients Poisson problem is formally identical to the one considered in this paper (with an appropriate redefinition of the metric coefficients). Another example is the determination of the geopotential in nonhydrostatic pressure coordinate models. For the latter, the elliptic operator has the form (Miller and White, 1984)

$$\nabla^2 h + \frac{\partial}{\partial p} \left( \frac{p^2}{H_0^2} \frac{\partial}{\partial p} \right)$$  \hspace{1cm} (32)

where $H_0$ is the base scale height of the fluid and $H_0^2$ the horizontal Laplacian. In this case, the relevant leptic ratio is clearly $\Delta_p/H_0$.

To conclude, we reiterate that when the approximate solution is used, regardless of the order, physically appropriate behavior is to be expected only at scales larger than the local depth. If smaller scales are present, e.g. to increase computational accuracy, careful consideration has to be given to the overall behavior of the scheme, as we have shown in the examples of the preceding sections. If a faithful representation of physics is needed at horizontal scales smaller than the total depth, then the original elliptic problem has to be solved exactly.

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**Appendix A**

In this paper, we have shown how it is possible to add the correct amount to dispersion at scales relevant to NLIWs propagation without incurring the full cost of a three-dimensional elliptic problem. Its implementation into a nonhydrostatic code should pose no serious difficulties. It would desirable to follow a similar program using a hydrostatic model as starting point. To do this, we need to distinguish between models that use the rigid-lid approximation and models that do not. For the former, the generation of corrections to the hydrostatic behavior is straightforward. In this case, calculation of the hydrostatic pressure involves a vertical quadrature and the solution of an horizontal elliptic problem to determine the “lid pressure”. As we have seen, this is equivalent in our scheme to solving Eq. (10) arresting the expansion to the lowest order. To generate the next order, the r.h.s. of the Poisson equation is corrected with the zeroth order solution, followed by a vertical integration and the solution of horizontal elliptic problem. Aside from trivial modification of the control algorithm, no new tools need to be coded, and the only price to pay is a manageable increase in the computational load.

Regrettably (from this point of view), most modern hydrostatic codes implement a free surface, where a Dirichlet condition for the pressure applies, and solve the fast vertically averaged barotropic mode separately from the slow internal baroclinic modes (Griffies, 2004, see e.g. p. 246–251). If we solve Eq. (10) with a Dirichlet condition at the surface using the expansion outlined in this paper, at the lowest order we obtain

$$p_h = \int_z^H \left( b \Delta \text{ds} \right) \simeq \int_z^H b \Delta \text{ds},$$  \hspace{1cm} (33)

where $\eta$ is the surface displacement, $H$ the mean surface height, $b$ the surface buoyancy and $\dot{b} = \dot{b} - b$ the deviation from the background buoyancy profile $b$. As before, the pressure and buoyancy are normalized with the reference density $\rho_0$, and for simplicity, we consider only small perturbations, which is justified when considering dispersion. Not surprisingly, this expression coincides with the pressure used in hydrostatic free-surface codes (Griffies, 2004, p. 57). To note here is the fact that the part dependent on the free surface has no $z$ dependency, and as such does not enter the equation for the baroclinic part. Conversely,

$$\int_0^H \int_z^H \text{dz} p_h = \int_z^H \int_z^H \text{dz} \Delta \text{ds} b' - \frac{1}{H^2} \int_0^N \int_z^H \text{dz} \int_z^H \text{dz} \Delta \text{ds} b'.$$  \hspace{1cm} (34)

which is the pressure affecting the baroclinic part, is identical to the expression derived in Eq. (17) for the rigid-lid case. Thus, for the baroclinic mode, the are no differences at the hydrostatic level between the rigid-lid and the free-surface cases, aside from the effect
of the vertical velocity generated by the horizontal divergence of the barotropic field on the buoyancy anomaly, whose strength is controlled by the Mach number \( M = N^2 H / g = O(10^{-2}) \), and thus negligible. Next we consider the first-order correction to the hydrostatic pressure (for simplicity, we consider a flat bottom),

\[
\frac{\partial^2}{\partial z^2} p_1 = -\nabla_\eta^2 p_0, \tag{35}
\]

\[
p_1 = \frac{1}{\nu} \left( \frac{H^2 - \nu^2 B}{2} + \int_0^H \int_0^L \int_0^L \int_0^L \dot{d} q \, \dot{d} b \int_0^H \int_0^L \int_0^L \int_0^L \nabla_\eta^2 \int_0^H \int_0^L \int_0^L \nabla_\eta^2 \right), \tag{36}
\]

with \( \nabla_\eta^2 \) denoting the horizontal Laplacian, and focus on the first term. Vertically averaged, returns a correction

\[
\frac{H^2}{3} \nabla_\eta^2 (\eta b) \tag{37}
\]

to the barotropic mode pressure, and a corresponding correction to the barotropic wave speed

\[
\frac{H^2 k^2}{6} \sqrt{gH}, \tag{38}
\]

which, as expected, extends the agreement with the fully hydrostatic speed to \( O(k^2 H^6) \) \( (\text{Kundu and Cohen, 2008, p. 240}) \). More interesting (and troubling) is the fact that, unlike the hydrostatic case, the surface displacement does not disappear when the vertically averaged component of \( p_1 \) is subtracted from it. Thus, the first-order correction, necessary to introduce dispersion, also tightly couples the baroclinic to the barotropic velocity, so that the latter must be solved on the fast time barotropic time scale. This, however, increases the computational load by a factor \( \mu^{-1} = O(100) \), a prohibitively expensive proposition!

A possible solution to this rather unpleasant prospectives is as follows: by construction, the baroclinic velocity field \( \mathbf{u} = (\mathbf{u}, \mathbf{v}, \mathbf{w}) \) does not alter the location of the free-surface. Thus, along a boundary surface \( \zeta \) (either at the bottom or at the surface),

\[
\dot{\mathbf{w}} = -\mathbf{u} \cdot \nabla_\eta \zeta = 0, \tag{39}
\]

where the last approximate equality holds in the linearized sense if \( \zeta \) is the surface. Furthermore, as we have seen earlier, at the hydrostatic level the pressure forcing appearing in the baroclinic field equation is the same regardless of the nature of the upper boundary. This suggests that the next order correction to the pressure be sought with Neumann conditions, essentially treating the problem of finding the higher-order corrections as if the baroclinic field be confined to a “frozen” rigid-lid like domain. The computational costs would be higher, but still manageable. There remain however a substantial coding issue. By its very nature, a free-surface code does not have an horizontal elliptic solver that can be “recycled” to calculate the higher order correction to the hydrostatic pressure distribution. Furthermore, the horizontal locality of the physics is exploited at several levels to improve efficiency, e.g. in how computational load is distributed among processors \( (\text{J. Wilkin, personal comm.}) \). This complicates the task of coding a nonlocal elliptic solver.

Thus, it is tempting to see if an appropriate Dirichlet boundary condition, which may depend on auxiliary prognostic equations, may be devised to avoid the horizontal elliptic problem when calculating \( p_1 \), just as the introduction of the free-surface in the hydrostatic case eliminates the need for an horizontal elliptic problem to calculate the lid pressure. To better appreciate this issue, it is necessary to delve further into the structure of the eigenmodes of the linearized free-surface equations. Let \( \mathbf{u} \) be the velocity field associated to an eigenmode, and consider its Helmholtz decomposition

\[
\mathbf{u} = \nabla \mathbf{\phi} + \nabla \times \mathbf{\psi}, \tag{40}
\]

where \( \mathbf{\phi} \) is the scalar and \( \mathbf{\psi} \) the vector potential \( (\text{Aris, 1989, p. 70}) \). Further, if we require that on the boundaries \( \mathbf{\psi} = 0 \) and use gauge invariance, we can ensure that

\[
\nabla \times \mathbf{u} = 0 \iff \mathbf{\psi} = 0, \tag{41}
\]

and

\[
\int_0^H \nabla_b \cdot (\nabla \times \mathbf{\psi}) \, dz = 0, \tag{42}
\]

\( \nabla_b \) being the horizontal divergence operator. That is, \( \nabla \times \mathbf{\psi} \) encodes the velocity field associated to the vorticity produced by the baroclinic torque acting within the fluid, and has no effect on the surface displacement, whereas \( \nabla \mathbf{\phi} \) encodes the irrotational component of the flow generated by the presence of the pliant surface. A straightforward calculation shows that the energy of the fast eigenmodes (whose speed is \( O(\sqrt{gH}) \)) is mostly contained in their irrotational component, i.e.

\[
\frac{||\nabla \times \mathbf{\psi}||^2}{||\nabla \mathbf{\phi}||^2} \ll 1. \tag{43}
\]

whereas the opposite is true for the slow eigenmodes (speed \( O(NH) \))

\[
\frac{||\nabla \mathbf{\phi}||^2}{||\nabla \times \mathbf{\psi}||^2} \ll 1. \tag{44}
\]

It is the irrotational component of the slow eigenmodes that needs to be isolated in order to provide an effective, slowly evolving, surface displacement to be used in Eq. \( (36) \) to calculate the correction to the hydrostatic pressure. The standard splitting technique (vertical integration) fails in this respect, because it folds the contribution of the irrotational component of fast and slow eigenmodes to surface displacement into a single entity, which has to be updated on the time scale of the fast eigenmodes. Thus, avoidance of the horizontal elliptic problem requires a way to separate the irrotational component of the slow eigenmodes from the irrotational component of the fast eigenmodes. Once this is done, two problems for the surface displacement can be written, each on its own time scale. The problem for the displacement associated to the fast eigenmodes can be safely decoupled from the inner vortical dynamics (as it is currently done), whereas the surface displacement due to the irrotational part of the slow eigenmodes has to be coupled to the internal vortical dynamics. How precisely this can be accomplished lies beyond the scope of this note.

To summarize, there are three possibilities to correct the hydrostatic pressure to add dispersion in a hydrostatic code. The first consists in imposing a Neumann condition on the pressure correction at the surface, and dealing with the ensuing horizontal elliptic problem, which has to be solved at each baroclinic time step. Numerically, this should not add too much to the overall cost, but requires a substantial coding investment unless an horizontal elliptic solver can be “recycled” from somewhere else in the code. The second consists in updating the entire system on the time scale of the fast mode. The position of the surface provides Dirichlet conditions on the pressure correction and avoids the horizontal elliptic problem. From a coding point of view, it requires relatively small modifications; however, the numerical cost is likely prohibitive. The third is to come up with a way to separate the irrotational component (in the sense described above) of the fast eigenmodes from the one of the slow eigenmodes, and proceed from them. Unfortunately, we are not in a position at the moment to give a recipe on how to do this.

**Appendix B**

Eq. \( (13) \) follows from

\[
|A - \sigma I| = -N^2 \frac{\partial^2}{\partial x^2} \int_0^L P \, dz + \sigma^2 b. \tag{45}
\]
Integrating vertically Eq. (10) leads to
\[ \frac{\partial^2}{\partial x^2} \int_0^z P \, dz = -B \frac{\partial P}{\partial z} \tag{46} \]
When used in Eq. (45) the above relation leads to
\[ \frac{N^2}{\sigma^2} \frac{\partial P}{\partial z} + b = 0 \tag{47} \]
and finally, taking the 2 derivative and using Eq. (10) again we arrive at the desired result.

References